Linear Integer Programming (ILP)

Aim – to formulate, solve by the *Solver*/Excel and graphically, integer and binary linear programming problems.

Definitions

An LP model is a ***pure ILP*** if all decision variables are required to have integer values – integrality constraints. If only some of the variables are defined as integer variables, it is called a ***mixed ILP***.

The ***linear relaxation*** of an ILP is the LP that arises by omitting the integrality constraints.

Parameters of the model:

c_j (j=1,2,…,n) coefficient of variable j at the OF;

b_i (i=1,2,…,m) right-hand-side of constraint i;

a_{ij} (i=1,2,…,m; j=1,2,…,n) technical coefficient.

Defining by x_j the level of the activity j ($j=1,\ldots,n$) - units of product j that should be produced-, and by Z the total performance (revenue), the corresponding models are:

Initial Problem

$$\text{(ILP)}\quad Z^* = Max\, Z = \sum_{j=1}^{n} c_j\, x_j$$

$$\text{s.to:} \begin{cases} \sum_{j=1}^{n} a_{ij}\, x_j \le b_i & i = 1,\ldots,m \\ x_j \ge 0 \ \text{ e inteiro} & j = 1,\ldots,n \end{cases}$$

Linear Relaxation

$$\text{(LR)}\quad Z_R^* = Max\, Z = \sum_{j=1}^{n} c_j\, x_j$$

$$\text{s.to:} \begin{cases} \sum_{j=1}^{n} a_{ij}\, x_j \le b_i & i = 1,\ldots,m \\ x_j \ge 0 & j = 1,\ldots,n \end{cases}$$

Property 1: Let Z^* be the optimum value of a maximization ILP and Z_R^* the optimum value of its linear relaxation, then: $Z^* \le Z_R^*$.

Graphical Resolution of an ILP with two decision variables

First solve the linear relaxation of the integer problem. If the integrality imposition is not satisfied by at least one variable, identify the feasible integer points (i.e., the integer points in the feasible region of the relaxed problem). Considering the feasible region of the integer problem and the objective function, determine, as usual, the optimal solution.

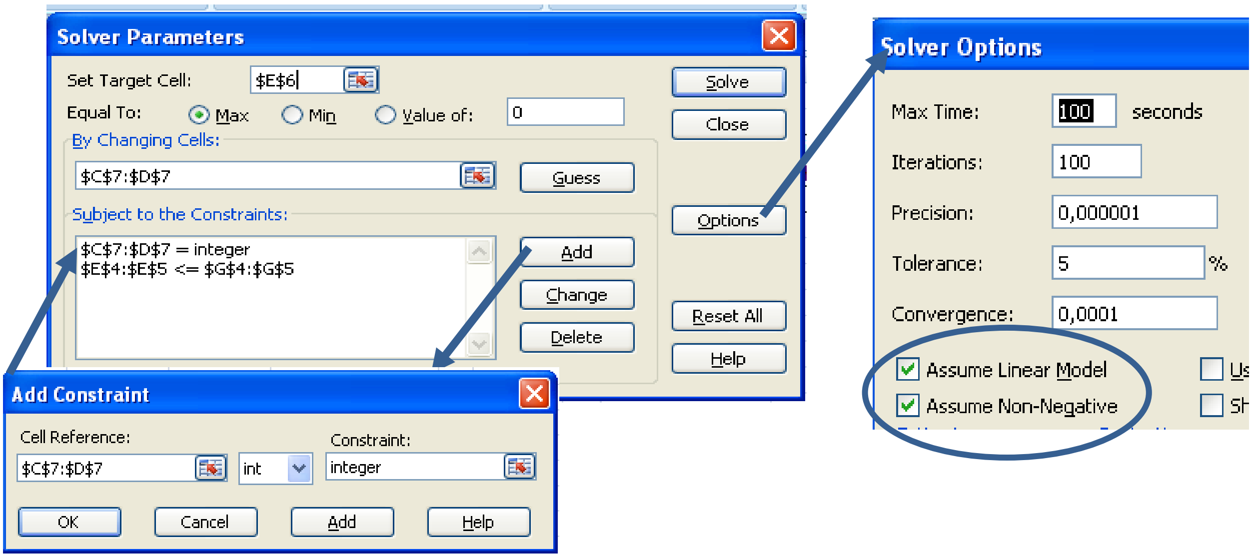
Resolution by the *Solver/Excel*

Exercise 48 – *TBA Airlines* – in an Excel spreadsheet, and similarly to an LP problem, write the problem to solve. Integrality constraints are included in the *Solver* parameters table.

	A	B	C	D	E	F	G	H
1								
2	TBA Airlines							
3			small	big			Disponibility	(Units)
4		Purchase Price	5	50	0	≤	100	millions of $
5		Max. Planes (N.er)	1	0	0	≤	2	
6		Annual Profit	1	5	0			
7		N.er of planes -	0	0				

	E
4	=SUMPRODUCT(C4:D4;$C$7:$D$7)
5	=SUMPRODUCT(C5:D5;$C$7:$D$7)
6	=SUMPRODUCT(C6:D6;$C$7:$D$7)

Solver – Indication of the target cell (E6), of the objective ("Maximization" or "Minimization"), and of the cells for decision variables values (C7:D7). Definition of the functional constraints ("Add"; E4:E5<=G4:G5) and of the integrality restrictions (C7:D7=integer).



Solution – Interpretation of the optimal solution, using the answer report provided by the *Solver* or using the corresponding *Excel* sheet.

	A	B	C	D	E	F	G	H
1								
2	TBA Airlines							
3			small	big			Disponibility	(Units)
4		Purchase Price	5	50	100	≤	100	millions of $
5		Max. Planes (N.er)	1	0	0	≤	2	
6		Annual Profit	1	5	10			
7		N.er of planes -	0	2				

Answer: Buy 2 big planes, with a profit of $10 *millions*. All the capital available is needed.

Formulation with Binary Variables

The use of binary variables is related with binary options, meaning decisions that only have two possibilities (yes or no). Binary variables are also useful to model alternative constraints, to characterize mutually exclusive or complementary products, or to assume fixed and variable costs/profits for the same product.

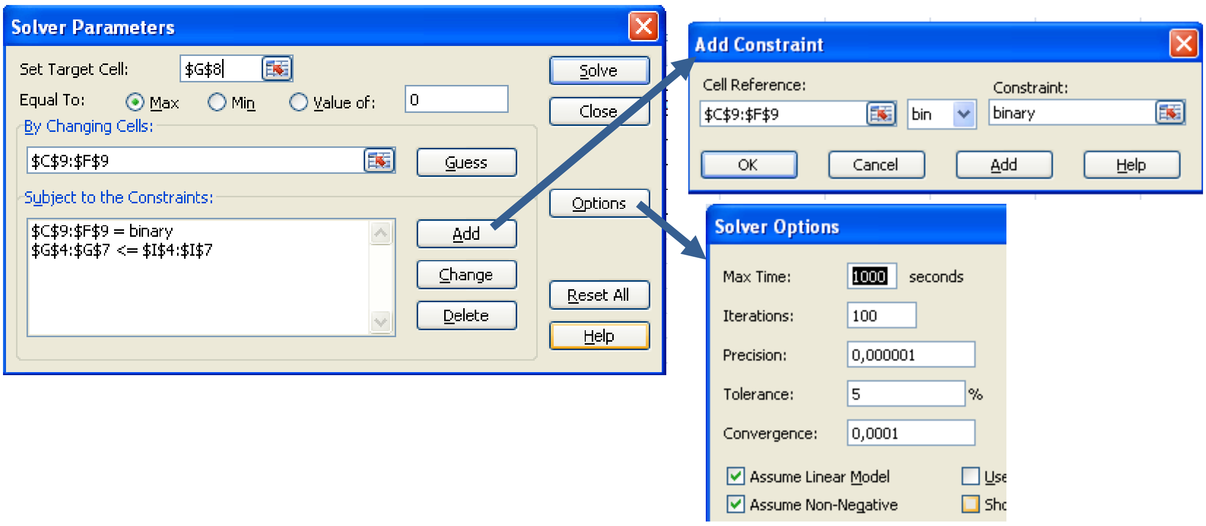
Binary Options – Resolution by the *Solver*/*Excel*

Prototype example – *California Manufacturing Company* – as usual, in an Excel sheet write down the problem to solve.

	B	C	D	E	F	G	H	I
2	California Manufactuting Co.							
3		Plant in LA	Plant in SF	Warehouse in LA	Warehouse in SF			
4	Capital required	6	3	5	2	0	≤	10
5	Only one warehouse	0	0	1	1	0	≤	1
6	LA-warehouse only if plant	-1	0	1	0	0	≤	0
7	SF-warehouse only if plant	0	-1	0	1	0	≤	0
8	Net Present Value	9	5	6	4	0		
9	Built	0	0	0	0			

	G
4	=SUMPRODUCT(C4:F4;$C$9:$F$9)
5	=SUMPRODUCT(C5:F5;$C$9:$F$9)
6	=SUMPRODUCT(C6:F6;$C$9:$F$9)
7	=SUMPRODUCT(C7:F7;$C$9:$F$9)
8	=SUMPRODUCT(C8:F8;$C$9:$F$9)

Solver – Indication of the target cell (G8), the objective ("Max" or "Min") and of the cells with the values of decision variables (C9:F9). "Add" the functional constraints (G4:G7<=I4:I7) and binary impositions (C9:F9=binary).



Solution – Interpretation of the solution using *Solver*/Excel outputs.

	A	B	C	D	E	F	G	H	I
2	California Manufactuting Co.								
3			Plant in LA	Plant in SF	Warehouse in LA	Warehouse in SF			
4		Capital required	6	3	5	2	9	≤	10
5		Only one warehouse	0	0	1	1	0	≤	1
6		LA-warehouse only if plant	-1	0	1	0	-1	≤	0
7		SF-warehouse only if plant	0	-1	0	1	-1	≤	0
8		Net Present Value	9	5	6	4	14		
9		Built	1	1	0	0			

Answer: Built only two plants, one in LA and the other in SF. The total revenue will be equal to \$14 *millions*.

Fixed-Charge Problem

Binary variables may be used to consider fixed charged or setup costs when undertaking an activity. Here binary variables are defined to indicate whenever the activity should be started.

Parameters of the Model:

Additionally to the usual parameters (c_j; b_i; a_{ij} with $j=1,\ldots,n$; $i=1,\ldots,m$), define F_j ($j=1,\ldots,n$) as the setup cost associated to the production of *j*.

Defining by *Z* the total performance, by x_j the units of product *j* ($j=1,\ldots,n$) to be to produced and the binary variables

$$y_j = \begin{cases} 1 & \text{if } j \text{ is produced} \\ 0 & \text{otherwise} \end{cases}$$

the ILP model is:

$$\text{(ILP)}\ \ Max\ \ Z = \sum_{j=1}^{n} c_j x_j - \sum_{j=1}^{n} F_j y_j \qquad \text{difference between revenue and set up cost,}$$

$$\text{s.to:} \begin{cases} \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad i=1,\ldots,m & \text{usual functional constraints,} \\ x_j \le M\ y_j \quad j=1,\ldots,n & \text{linking constraints,} \\ x_j \ge 0 \quad j=1,\ldots,n & \text{variables definition,} \\ y_j \in \{0,1\} \quad j=1,\ldots,n & \end{cases}$$

where *M* is an extremely large positive number.

Note that linking constraints, relating binary and correspondent initial decision variables, are needed to guarantee that a product may be produced ($x_j>0$) only if the correspondent set up cost was paid ($y_j=1$). A positive value for *M* must be chosen bearing in mind that the values of each decision variable x_j should not be limited by these linking constraints.

Prototype example (chap. 1) – *WYNDOR GLASS CO.* – suppose that set up costs of 7 *m.u.* and of 13 *m.u.*, respectively, should be associated to the production of doors and windows. Define:

$$y_j = \begin{cases} 1 & \text{if } j \text{ is produced} \\ 0 & \text{if not} \end{cases} \quad (j=1,2)$$

where $j=1$ respects to the doors and $j=2$ to the windows. As before, let the decision variables x_j ($j=1,2$) represent the number of batches of *j* produced. The model is:

$$Max\ Z = 3x_1 + 5x_2 - 7y_1 - 13y_2$$

$$\text{s.to: } \begin{cases} \left.\begin{array}{l} x_1 \le 4 \\ 2x_2 \le 12 \\ 3x_1 + 2x_2 \le 18 \end{array}\right\rangle \text{usual funcional constraints} \\ \left.\begin{array}{l} x_1 \le M\ y_1 \\ x_2 \le M\ y_2 \end{array}\right\rangle \text{linking constraints} \\ x_1, x_2 \ge 0 \\ y_1, y_2 \in \{0,1\} \end{cases}$$

where, e.g., M=1000.

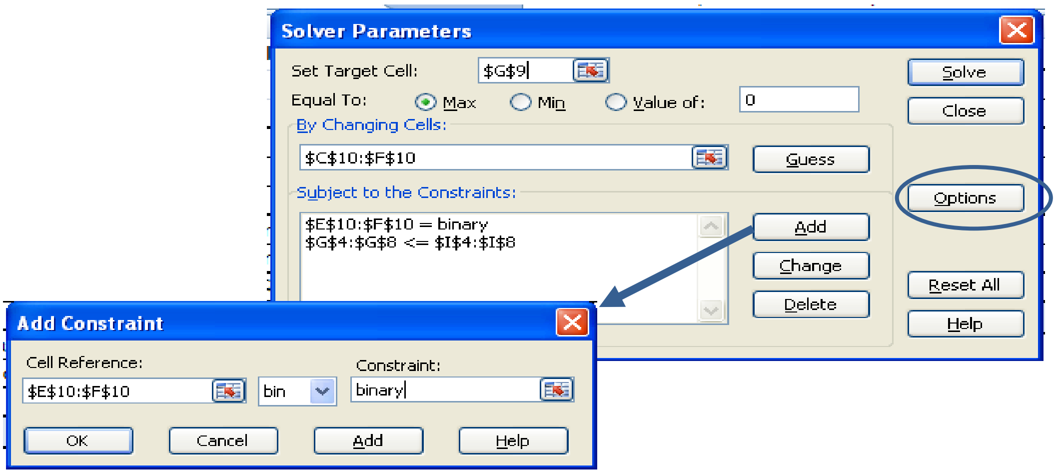
Resolution by the *Solver/Excel*

Prototype example (chap. 1) – *Wyndor Glass Co.* – in an Excel sheet write the problem to solve, where the linking constraints should be in the equivalent form $x_j - My_j \le 0$.

	A	B	C	D	E	F	G	H	I
1	Wyndor Glass Co. with set up costs				It should be produced				
2			Hours used per batch produced		doors?	windows?			Hours Available
3			of doors	of windows	y1	y2			
4		Plant 1 (m-h)	1	0	0	0	0	≤	4
5		Plant 2 (m-h)	0	2	0	0	0	≤	12
6		Plant 3 (m-h)	3	2	0	0	0	≤	18
7		Linking constraints (doors)	1	0	-1000	0	0	≤	0
8		Linking constraints (windows)	0	1	0	-1000	0	≤	0
9		Profit	3	5	-7	-13	0		
10		Batches produced	0	0	0	0			

	G
4	=SUMPRODUCT(C4:F4;C$10:F$10)
5	=SUMPRODUCT(C5:F5;C$10:F$10)
6	=SUMPRODUCT(C6:F6;C$10:F$10)
7	=SUMPRODUCT(C7:F7;C$10:F$10)
8	=SUMPRODUCT(C8:F8;C$10:F$10)
9	**=SUMPRODUCT(C9:F9;C$10:F$10)**

Solver – Indication of the target cell (G9), of the objective ("Max" or "Min") and of the cells with the values for variables (C10:F10). "Add" the functional and linking constraints (G4:G8<=I4:I8) and the binary impositions (E10:F10=binary).



Solution – Interpretation of the solution using *Solver*/Excel outputs.

	A	B	C	D	E	F	G	H	I
1	Wyndor Glass Co. with set up costs				It should be produced				
2			Hours used per batch produced		doors?	windows?			Hours Available
3			of doors	of windows	y1	y2			
4		Plant 1 (m-h)	1	0	0	0	0	≤	4
5		Plant 2 (m-h)	0	2	0	0	12	≤	12
6		Plant 3 (m-h)	3	2	0	0	12	≤	18
7		Linking constraints (doors)	1	0	-1000	0	0	≤	0
8		Linking constraints (windows)	0	1	0	-1000	-994	≤	0
9		Profit	3	5	-7	-13	17		
10		Batches produced	0	6	0	1			

Answer: Only windows should be produced, 6 batches (x_2=6; y_2=1). The total profit is equal to 17 *m.u.*.

Mutually Exclusive Products

Binary variables are also useful to impose a maximum number of activities, or to consider activities that should not be developed simultaneously.

Parameters of the model:

Consider the usual parameters c_j; b_i; a_{ij} , where $j=1,\ldots,n$; $i=1,\ldots,m$.

Suppose that no more than de $\boldsymbol{K}<n$ products should be produced and that both products r and s are mutually exclusive, meaning that they cannot be produced simultaneously.

Defining by Z the total performance (revenue), by x_j the level of the activity j ($j=1,\ldots,n$), and the binary variables:

$$y_j = \begin{cases} 1 & \text{if } j \text{ is produced} \\ 0 & \text{otherwise} \end{cases}$$

the integer linear programming model is:

$$\text{(ILP)}\quad Max\, Z = \sum_{j=1}^{n} c_j\, x_j$$

$$\text{s.to:} \begin{cases} \sum_{j=1}^{n} a_{ij}\, x_j \le b_i \quad i=1,\ldots,m & \text{usual functional constraints,} \\ x_j \le M\, y_j \quad j=1,\ldots,n & \text{linking constraints,} \\ \sum_{j=1}^{n} y_j \le K & \text{no more than } K \text{ products,} \\ y_r + y_s \le 1 & r \text{ and } s \text{ are mutually exclusive,} \\ x_j \ge 0 \quad j=1,\ldots,n & \\ y_j \in \{0,1\} \quad j=1,\ldots,n & \text{variables definition,} \end{cases}$$

where M is an extremely large positive number.

The Solver/Excel model and resolution are similar to the ones presented before.

Either-Or Constraints

Alternative (either-or) constraints may easily be defined with binary variables indicating witch constraint should be imposed.

Parameters of the model:

Consider the usual parameters c_j; b_i; a_{ij} , where $j=1,\ldots,n$; $i=1,\ldots,m$.

Suppose that a choice should be made between two constraints, so that only one (R1 or R2) must hold. The other can hold, but is not required to do so.

Defining by Z the total performance (revenue), by x_j the level of the activity j ($j=1,\ldots,n$), and the binary variables, for k=1, 2:

$$y_k = \begin{cases} 1 & \text{if restriction Rk holds} \\ 0 & \text{if Rk is not required.} \end{cases}$$

the integer linear programming model is:

$$\text{(ILP)}\quad Max\, Z = \sum_{j=1}^{n} c_j\, x_j$$

$$\text{s.to:}\ \begin{cases} \sum_{j=1}^{n} a_{1j}\, x_j \le b_1 + M\,(1-y_1) & \text{(R1)} \\ \sum_{j=1}^{n} a_{2j}\, x_j \le b_2 + M\,(1-y_2) & \text{(R2)} \\ y_1 + y_2 = 1 & \text{(a)} \\ \sum_{j=1}^{n} a_{ij}\, x_j \le b_i \quad i=3,\ldots,m & \\ x_j \ge 0 \quad j=1,\ldots,n & \\ y_k \in \{0,1\} \quad k=1,2 & \end{cases}$$

(R1), (R2): either-or constraints,
(a): only one is mandatory,
usual functional constraints,
variables definition.

Being M an extremely large positive number, it follows that:

If $y_1 = 1$	If $y_1 = 0$
$\begin{cases} \text{(a)} \Rightarrow y_2 = 0 \\ \text{(R1)} \Rightarrow \sum_{j=1}^{n} a_{1j}\, x_j \le b_1 \quad \text{holds} \\ \text{(R2)} \Rightarrow \sum_{j=1}^{n} a_{2j}\, x_j \le b_2 + M \quad \text{redundant} \end{cases}$	$\begin{cases} \text{(a)} \Rightarrow y_2 = 1 \\ \text{(R1)} \Rightarrow \sum_{j=1}^{n} a_{1j}\, x_j \le b_1 + \text{M} \quad \text{redundant} \\ \text{(R2)} \Rightarrow \sum_{j=1}^{n} a_{2j}\, x_j \le b_2 \quad \text{holds.} \end{cases}$

Thus, only the most attractive constraint is then chosen to be an active part of the model.

Prototype example (chap. 1) – *WYNDOR GLASS CO.* – suppose that Plant 3 may be replaced by other plant (Plant 4) not yet considered. Plant 4 has an available capacity of 52 hours per week. In this plant, the production of one batch of doors uses 3h, while the windows require 8h. Defining, additionally, and for k=3, 4:

$$y_k = \begin{cases} 1 & \text{if restriction Rk holds} \\ 0 & \text{if Rk is not required,} \end{cases}$$

the new ILP model became:

$$Max\ Z = 3x_1 + 5x_2$$

$$\text{s.to: } \begin{cases} x_1 \leq 4 \\ 2x_2 \leq 12 \\ 3x_1 + 2x_2 \leq 18 + M(1 - y_3) & \text{Plant3} \\ 3x_1 + 8x_2 \leq 52 + M(1 - y_4) & \text{Plant4}, \quad \text{where, e.g., } M=1000. \\ y_3 + y_4 = 1 \\ x_1, x_2 \geq 0 \\ y_3, y_4 \in \{0,1\} \end{cases}$$

Graphical Resolution

The graphical resolution of a problem with two decision variables and with binary variables to model alternative constraints starts with the graphical representation of the usual functional and nonnegative constraints. In the example, one should first represent the region defined by:

$$\begin{cases} x_1 \leq 4 \\ 2x_2 \leq 12 \\ x_1, x_2 \geq 0 \end{cases}$$

Hence, consider the union between the either-or restrictions ($3x_1 + 2x_2 \leq 18 \vee 3x_1 + 8x_2 \leq 52$), intercepted with the region first represented. Note that, in these models, the feasible region may be depicted by a non-convex set, as the case of this example. The objective function should then be drawn to find out the optimal solution.

Resolution by the *Solver*/Excel

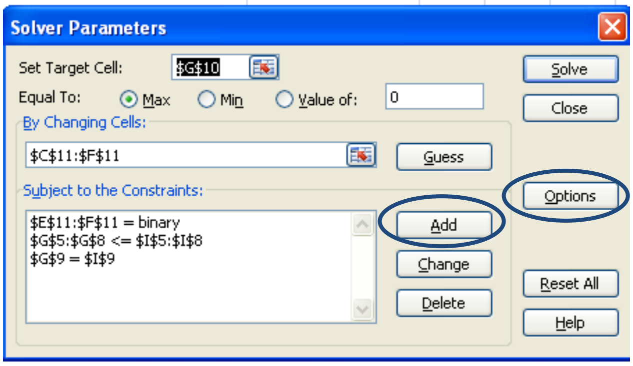
Prototype example – *WYNDOR GLASS CO.* – write down the problem to solve in an Excel sheet.

	A	B	C	D	E	F	G	H	I
1	**Either-Or Constraints**								
2						M=	1000		
3			Hours used per batch produced						Hours Available
4			of doors	of windows	y3	y4	Total		
5		Plant 1 (m-h)	1	0	0	0	0	≤	4
6		Plant 2 (m-h)	0	2	0	0	0	≤	12
7	Or Plant 3	Plant 3 (m-h)	3	2	1000	0	0	≤	1018
8	Or Plant 4	Plant 4 (m-h)	3	8	0	1000	0	≤	1052
9	impose only one		0	0	1	1	0	=	1
10		Profit	3	5	0	0	0		
11	Batches produced		0	0	0	0			

	I
7	=18+G2
8	=52+G2

usual formulas

Solver – Indication of the target cell (G10), of the objective ("Max" or "Min") and of the cells with the values for variables (C11:F11). "Add" the usual functional (G5:G6<=I5:I6), the alternative (G7:G8<=I7:I8; G9=I9) and the binary constraints (E11:F11=binary).



Solution – Interpretation of the solution using *Solver*/Excel outputs.

	A	B	C	D	E	F	G	H	I
1	**Either-Or Constraints**								
2						M=	1000		
3			Hours used per batch produced						Hours Available
4			of doors	of windows	y3	y4	Total		
5		Plant 1 (m-h)	1	0	0	0	4	≤	4
6		Plant 2 (m-h)	0	2	0	0	10	≤	12
7	Or Plant 3	Plant 3 (m-h)	3	2	1000	0	22	≤	1018
8	Or Plant 4	Plant 4 (m-h)	3	8	0	1000	1052	≤	1052
9	impose only one		0	0	1	1	1	=	1
10		Profit	3	5	0	0	37		
11	Batches produced		4	5	0	1			

Answer: It should be produced 4 batches of doors and 5 of windows, per week. For this purpose, 4*h* of Plant 1 are needed (all of its available capacity), 10*h* of Plant 2, and 52*h* of Plant 4 (and y_4=1). Plant 3 is not used. The total profit is 37*m.u.*.